

COVARIANT AFFINE INTEGRAL QUANTIZATION(S)

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ABSTRACT. Covariant affine integral quantization of the half-plane is studied and applied to the motion of a particle on the half-line. We examine the consequences of different quantizer operators built from weight functions on the half-plane. To illustrate the procedure, we examine two particular choices of the weight function, yielding thermal density operators and affine inversion respectively. The former gives rise to a temperature-dependent probability distribution on the half-plane whereas the later yields the usual canonical quantization and a quasi-probability distribution (affine Wigner function) which is real, marginal in both momentum p and position q .

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1. INTRODUCTION

This work is a continuation of previous studies devoted to affine integral quantization of the half-plane [1, 2, 3, 4] and its applications to early or quantum cosmology [5, 6, 7, 8, 9, 10]. In the latter works, the method was based on the use of affine coherent states or wavelets. Here we extend it to a class of operators including density operators and the inversion operator allowing a sort of Wigner transform for the affine group.

The coherent state quantization which was used in the applications cited above is a particular approach pertaining to what is named in [1] integral quantization. When a group action is involved in the construction, one can insist on covariance aspects of the method. A detailed presentation of the procedure is given in [1] and in Chapt. 11 of [2]. In the case of the Weyl-Heisenberg group, weight functions defined on the euclidean plane viewed as a phase space were at the heart of the construction of the covariant integral quantization. In a certain sense these functions, or their symplectic Fourier transforms, correspond to the Cohen “ f ” function [11] (for more details see [12] and references therein) or to Agarwal-Wolf filter functions [13], even though these authors were not directly proposing new quantization procedures.

The organisation of the paper is as follows. In Section 2, we give a short compendium of covariant integral quantization approach. All necessary details on the affine group of the real line are given in Section 3, geometry, unitary irreducible representation (UIR), UIR matrix elements in a particular Laguerre Hilbertian basis from which is built a specific density operator called thermal state. In Section 4 we show how to build symmetric operators as UIR transform of weight functions on the half-plane. The affine transports of these operators allow a resolution of the identity, a necessary ingredient for quantization. The main results concerning these affine covariant quantizations are presented in Section 5. In Section 6 we particularise in Subsection 6.1 this quantization method by using one-rank density operators, i.e. affine coherent states (ACS), recalling results given in previous articles, and we also consider in Subsection 6.2 the quantization issued from Laguerre thermal states. Section 7 is devoted to the quantization issued from the affine inversion operator which correspond to a specific weight function and which leads to a Wigner-like quasi-distribution on the half-plane. To a certain extent, the results described in this section are related to previous works, particularly the pioneering contributions by J. Bertrand and P. Bertrand [14, 15], the more mathematically oriented articles by Ali *et al* [16, 17] and by the comprehensive approach by Gayral *et al* [18] with references therein, devoted to Fourier analysis on the (full) affine group, its Stratonovich–Weyl quantizer, and the corresponding Wigner functions. The application to the half-oscillator is given as an elementary example and graphical illustrations are given to compare the Wigner quasi-distribution with probability distributions issued from ACS. For another interesting approach, based on representations of the larger group $SL(2, \mathbb{R})$ and its covering(s), to the quantization of the motion of a particle moving on a half-line with ‘hard wall’ boundary condition, see the recent PhD thesis of Jung [19]. We conclude in Section 8 with comments and a brief description of our future work on the same subject.

2. COVARIANT INTEGRAL QUANTIZATIONS

Lie group representations [20] offers a wide range of possibilities for implementing integral quantization(s). Let G be a Lie group with left Haar measure $d\mu(g)$, and let $g \mapsto U(g)$ be a unitary irreducible representation (UIR) of G in a Hilbert space \mathcal{H} . Let us consider a bounded operator M on \mathcal{H} and suppose that the operator

$$(2.1) \quad R := \int_G M(g) d\mu(g), \quad M(g) := U(g) M U(g)^\dagger,$$

is defined in a weak sense. From the left invariance of $d\mu(g)$ we have

$$(2.2) \quad U(g_0) R U^\dagger(g_0) = \int_G M(g_0 g) d\mu(g) = R,$$

so M commutes with all operators $U(g)$, $g \in G$. Thus, from Schur's Lemma, $M = c_M I$ with

$$(2.3) \quad c_M = \int_G \text{tr}(\rho_0 M(g)) d\mu(g),$$

where the unit trace non-negative operator (\sim density operator) ρ_0 is chosen in order to make the integral converge. This family of operators provides the resolution of the identity on \mathcal{H} .

$$(2.4) \quad \int_G M(g) d\nu(g) = I, \quad d\nu(g) := \frac{d\mu(g)}{c_M}.$$

and the subsequent quantization of complex-valued functions (or distributions, if well-defined) on G

$$(2.5) \quad f \mapsto A_f = \int_G M(g) f(g) d\nu(g),$$

This linear map, function \mapsto operator in \mathcal{H} , is covariant in the sense that

$$(2.6) \quad U(g) A_f U(g)^\dagger = A_{\mathfrak{U}_r(g)f}.$$

In the case when $f \in L^2(G, d\mu(g))$, the action $(\mathfrak{U}_r(g)f)(g') := f(g^{-1}g')$ defines the regular representation of G .

The other face of integral quantization concerns a consistent semi-classical analysis of the operator A_f based on it. It is implemented through the study of the so-called lower symbols. Suppose that M is a density operator $M = \rho$ on \mathcal{H} . Then the operators $\rho(g)$ are also density, and this allows to build a new function $\check{f}(g)$, called lower symbol, as

$$(2.7) \quad \check{f}(g) \equiv \check{A}_f := \int_G \text{tr}(\rho(g) \rho(g')) f(g') d\nu(g').$$

The map $f \mapsto \check{f}$ is a generalization of the Berezin or heat kernel transform on G (see [21] and references therein). Choosing for M a density operator ρ has multiple advantages, peculiarly in regard to probabilistic aspects both on classical and quantum levels [4].

Let us illustrate the above procedure with the case of square integrable UIR's and rank one ρ . For a square-integrable UIR U for which $|\psi\rangle$ is an admissible unit vector, i.e.,

$$(2.8) \quad c(\psi) := \int_G d\mu(g) |\langle \psi | U(g) | \psi \rangle|^2 < \infty,$$

the resolution of the identity is obeyed by the coherent states $|\psi_g\rangle = U(g)|\psi\rangle$, in a generalized sense, for the group G :

$$(2.9) \quad \int_G \rho(g) d\nu(g) = I, \quad d\nu(g) = \frac{d\mu(g)}{c(\psi)}, \quad \rho(g) = |\psi_g\rangle \langle \psi_g|.$$

3. THE AFFINE GROUP AND ITS REPRESENTATION U

3.1. Half-plane and affine group. The half-plane can be viewed as the phase space for the (time) evolution a positive physical quantity such as the position of a particle moving in the half-line, or, at the opposite, the contracting-expanding volume of the Universe. Let the upper half-plane $\Pi_+ := \{(q, p) | p \in \mathbb{R}, q > 0\}$ be equipped with the measure $dqdp$. Together with the multiplication

$$(3.1) \quad (q, p)(q_0, p_0) = (qq_0, p_0/q + p), \quad q \in \mathbb{R}_+^*, p \in \mathbb{R},$$

the unity $(1, 0)$ and the inverse

$$(3.2) \quad (q, p)^{-1} = \left(\frac{1}{q}, -qp \right),$$

Π_+ is viewed as the affine group $\text{Aff}_+(\mathbb{R})$ of the real line, and the measure $dqdp$ is left-invariant with respect to this action.

3.2. Representation. The affine group $\text{Aff}_+(\mathbb{R})$ has two non-equivalent UIR U_\pm [22, 23]. Both are square integrable and this is the rationale behind *continuous wavelet analysis* [24, 25, 26, 2]. Only the UIR $U_+ \equiv U$ is concerned in the rest of the paper. This representation is realized in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+^*, dx)$ as

$$(3.3) \quad U(q, p)\psi(x) = (e^{ipx}/\sqrt{q})\psi(x/q).$$

The most immediate (and well-known) orthonormal basis of $L^2(\mathbb{R}_+^*, dx)$ is that one which is built from Laguerre polynomials,

$$(3.4) \quad e_n^{(\alpha)}(x) := \sqrt{\frac{n!}{(n+\alpha)!}} e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} L_n^{(\alpha)}(x), \quad \int_0^\infty e_n^{(\alpha)}(x) e_{n'}^{(\alpha)}(x) dx = \delta_{nn'},$$

where $\alpha > -1$ is a free parameter, and $(n+\alpha)! = \Gamma(n+\alpha+1)$. Actually, since we wish to work with functions which, with a certain number of their derivatives, vanish at the origin, the parameter α should be imposed to be larger than some $\alpha_0 > 0$.

The matrix elements $U_{mn}^{(\alpha)}(q, p) := \langle e_m^{(\alpha)} | U(q, p) | e_n^{(\alpha)} \rangle$ of the representation U with respect to this basis are given in terms of Gauss hypergeometric polynomial or Jacobi

polynomial [27] by

(3.5)

$$\begin{aligned}
 U_{mn}^{(\alpha)}(q, p) &= \frac{1}{q^{(\alpha+1)/2}} \sqrt{\frac{m!n!}{(m+\alpha)!(n+\alpha)!}} \int_0^\infty dx e^{-\left(\frac{1}{2} + \frac{1}{2q} - ip\right)x} x^\alpha L_m^{(\alpha)}(x) L_n^{(\alpha)}\left(\frac{x}{q}\right) \\
 &= 2^{\alpha+1} \sqrt{\binom{m+n+\alpha}{m} \binom{m+n+\alpha}{n}} q^{(\alpha+1)/2} \frac{Z_-^m \bar{Z}_-^n}{\bar{Z}_+^{m+n+\alpha+1}} \times \\
 &\quad \times {}_2F_1\left(-m, -n; -m-n-\alpha; \left|\frac{Z_+}{Z_-}\right|^2\right) \\
 (3.6) \quad &= 2^{\alpha+1} \sqrt{\frac{(n+\alpha)!}{(m+\alpha)!} \frac{m!}{n!}} q^{(\alpha+1)/2} \frac{\bar{Z}_-^{n-m} Z_+^m}{\bar{Z}_+^{n+\alpha+1}} P_m^{(n-m, \alpha)}(Y),
 \end{aligned}$$

where

$$(3.7) \quad Z_\pm := q \pm 1 + 2iqp, \quad Y := 1 - 2 \left| \frac{Z_-}{Z_+} \right|^2.$$

The integral formula involving associated Laguerre polynomials is found in [28], 850-4. One easily verifies the unitarity

$$(3.8) \quad U_{mn}^{(\alpha)}(q^{-1}, -qp) = U_{mn}^{(\alpha)-1}(q, p) = (U^{(\alpha)})_{mn}^\dagger(q, p) = \overline{U_{nm}^{(\alpha)}(q, p)}$$

from $Y(q, p) = Y(1/q, -qp)$ and [27]

$$P_m^{(n-m, \alpha)}(X) = \frac{(m+\alpha)!}{(n+\alpha)!} \frac{n!}{m!} \left(\frac{X-1}{2} \right)^{m-n} P_n^{(m-n, \alpha)}(X).$$

These matrix elements obey orthogonality relations for the affine group. Since this group is not unimodular, there exists a positive self-adjoint and invertible operator \mathbb{C}_{DM} , the Duflo-Moore operator, such that [2]

$$(3.9) \quad \int_{\Pi_+} dq dp \langle U(q, p) \psi | \phi \rangle \overline{\langle U(q, p) \psi' | \phi' \rangle} = \langle \mathbb{C}_{\text{DM}} \psi | \mathbb{C}_{\text{DM}} \psi' \rangle \langle \phi' | \phi \rangle,$$

for any pair (ψ, ψ') of admissible vectors, i.e. which obey $\|\mathbb{C}_{\text{DM}} \psi\| < \infty$, $\|\mathbb{C}_{\text{DM}} \psi'\| < \infty$, and any pair (ϕ, ϕ') of vectors in $L^2(\mathbb{R}_+^*, dx)$. For the affine group, the Duflo-Moore operator is the multiplication operator

$$(3.10) \quad \mathbb{C}_{\text{DM}} \psi(x) := \sqrt{\frac{2\pi}{x}} \psi(x) \equiv \sqrt{\frac{2\pi}{Q}} \psi(x),$$

where $Q\psi(x) := x\psi(x)$ is the basic positive self-adjoint multiplication operator with $|x\rangle = \delta_x$ as eigendistributions. Thus, the admissibility condition for $\psi \in L^2(\mathbb{R}_+^*, dx)$ amounts to

$$(3.11) \quad \|\mathbb{C}_{\text{DM}} \psi\|^2 = 2\pi \int_0^{+\infty} \frac{dx}{x} |\psi(x)|^2 < \infty.$$

For the sequel, due to the non-unimodular nature of the affine group, that operator \mathbb{C}_{DM} is expected to play an important rôle. A first important and direct consequence of Eq (3.9) is the resolution of the identity in $L^2(\mathbb{R}_+^*, dx)$:

$$(3.12) \quad \frac{1}{\|\mathbb{C}_{\text{DM}}\psi\|^2} \int_{\Pi_+} dq dp U(q, p) |\psi\rangle \langle \psi| U^\dagger(q, p) = I.$$

satisfied by the family of *affine coherent states* (ACS) $U(q, p)|\psi\rangle \equiv |q, p\rangle$ built from the admissible vector ψ through the unitary affine transport U .

Note that the equation (3.9) implies for the matrix elements the integral formula,

$$(3.13) \quad \int_{\Pi_+} dq dp \overline{U_{mn}^{(\alpha)}(q, p)} U_{m'n'}^{(\alpha)}(q, p) = 2\pi \delta_{mm'} \langle e_n^{(\alpha)} | (1/Q) e_{n'}^{(\alpha)} \rangle.$$

Let us end this section with a formula giving the trace of the operator $U(q, p)$ from [27].

$$(3.14) \quad \text{Tr } U(q, p) = \sum_{m=0}^{\infty} U_{mm}^{(\alpha)}(q, p) = \frac{\sqrt{q}}{|q-1|}.$$

3.3. The “thermal state” case. Here we consider the temperature-dependent density operator

$$(3.15) \quad \rho_t = (1-t) \sum_{n=0}^{\infty} t^n |e_n\rangle \langle e_n|, \quad t = e^{-\frac{\hbar\omega}{k_B T}},$$

where $\{|e_n\rangle | n \in \mathbb{N}\}$ is an orthonormal basis of \mathcal{H} . By choosing the Laguerre orthonormal basis (3.4) for which $\rho_t \equiv \rho_t^{(\alpha)}$, we find from [27] that the operator ρ_t acts on $\mathcal{H} = L^2(\mathbb{R}_+^*, dx)$ as the integral transform

$$(3.16) \quad \rho_t^{(\alpha)} : \psi(x) \mapsto \rho_t^{(\alpha)}(\psi)(x) = \int_0^\infty \mathcal{K}_t^{(\alpha)}(x, y) \psi(y) dy,$$

where the integral kernel is given by

$$(3.17) \quad \mathcal{K}_t^{(\alpha)}(x, y) = t^{-\alpha/2} e^{-\frac{1}{2} \frac{1+t}{1-t}(x+y)} I_\alpha \left(2 \frac{\sqrt{txy}}{1-t} \right).$$

This closed form is obtained from the expression of the Poisson kernel for the Laguerre polynomials in terms of a modified Bessel function, 8.976–1 in [28]

$$(3.18) \quad \sum_{n=0}^{\infty} \frac{n!}{(n+\alpha)!} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) t^n = \frac{1}{(xyt)^{\alpha/2} (1-t)} e^{-(x+y) \frac{t}{1-t}} I_\alpha \left(\frac{2\sqrt{xyt}}{1-t} \right),$$

for $|t| < 1$. Again, one derives from the Schur lemma that

$$(3.19) \quad \rho_t^{(\alpha)}(q, p) := U(q, p) \rho_t^{(\alpha)} U(q, p)^\dagger$$

resolves the identity,

$$(3.20) \quad \int_{\Pi_+} \rho_t^{(\alpha)}(q, p) \frac{dq dp}{c_{\rho_t^{(\alpha)}}} = I,$$

where the constant $c_{\rho_t^{(\alpha)}}$ is obtained through standard calculations in wavelet theory,

$$\begin{aligned}
c_{\rho_t^{(\alpha)}} &= \int_{\Pi_+} \langle e_0^{(\alpha)} | \rho_t(q, p) | e_0^{(\alpha)} \rangle dq dp \\
&= (1-t) \sum_{n=0}^{\infty} t^n \int_{\Pi_+} \langle e_0^{(\alpha)} | U(q, p) | e_n^{(\alpha)} \rangle \langle e_n^{(\alpha)} | U(1/q, -qp) \rangle dq dp \\
&= 2\pi(1-t) \sum_{n=0}^{\infty} \frac{n!}{\alpha!(n+\alpha)!} t^n \int_0^{\infty} dq \int_0^{\infty} dx e^{-(1+q)x} x^{2\alpha} q^{\alpha} \left(L_n^{(\alpha)}(x) \right)^2 \\
&= 2\pi(1-t) \int_0^{\infty} \frac{dx}{x} e^{-x} x^{\alpha} \sum_{n=0}^{\infty} \frac{n!}{(n+\alpha)!} t^n \left(L_n^{(\alpha)}(x) \right)^2 \\
&= 2\pi t^{-\alpha/2} \int_0^{\infty} \frac{dx}{x} e^{-\frac{1+t}{1-t}x} I_{\alpha} \left(\frac{2\sqrt{t}}{1-t} x \right) \\
(3.21) \quad &= \frac{2\pi}{\alpha}.
\end{aligned}$$

Here we have used the integral formula involving a Bessel function:

$$(3.22) \quad \int_0^{\infty} \frac{dx}{x} e^{-\gamma x} I_{\alpha}(\mu x) = \frac{1}{\alpha} \left[\frac{\gamma}{\mu} - \sqrt{\frac{\gamma^2}{\mu^2} - 1} \right]^{\alpha}.$$

Thus, the resolution of the identity imposes the painless restriction $\alpha > 0$ and reads finally

$$(3.23) \quad \alpha \int_{\Pi_+} \rho_t^{(\alpha)}(q, p) \frac{dq dp}{2\pi} = I.$$

4. SYMMETRIC OPERATORS FROM WEIGHT FUNCTIONS

A general method to get density or more general (symmetric) bounded operators is the following. Let us choose like in [3] (see also [1] for the Weyl-Heisenberg group) a suitably localized weight function $\varpi(q, p)$ on the half-plane such that the integral

$$(4.1) \quad \int_{\Pi_+} C_{DM}^{-1} U(q, p) C_{DM}^{-1} \varpi(q, p) dq dp := M^{\varpi}$$

defines a symmetric operator. The non-unimodularity of the affine group justifies the double presence of the inverse of the Duflo-Moore operator, at the difference of the situation we had for the Weyl-Heisenberg group in [1]. From the symmetric property $M^{\varpi} = M^{\varpi^\dagger}$, we find that the weight function must satisfy

$$(4.2) \quad \varpi(q, p) = \frac{1}{q} \overline{\varpi\left(\frac{1}{q}, -qp\right)}.$$

Trivial (but with not so trivial consequences!) solutions are

$$(4.3) \quad \varpi(q, p) = \frac{1}{\sqrt{q}},$$

$$(4.4) \quad \varpi(q, p) = e^{\pm i\sqrt{q}p}.$$

These two elementary solutions combine to yield the self-adjoint nilpotent inversion operator \mathcal{I} defined on $L^2(\mathbb{R}_+^*, dx)$:

$$(4.5) \quad \int_{\Pi_+} \mathbf{C}_{\text{DM}}^{-1} U(q, p) \mathbf{C}_{\text{DM}}^{-1} \frac{e^{-i\sqrt{q}p}}{2\sqrt{q}} dq dp = \mathcal{I}, \quad (\mathcal{I}\psi)(x) := \frac{1}{x} \psi\left(\frac{1}{x}\right), \quad \mathcal{I}^2 = I.$$

Note here that the two inverse Dufflo-Moore operators simplify to the factor $1/(2\pi)$ since $1/\sqrt{Q} \mathcal{I} 1/\sqrt{Q} = \mathcal{I}$. Section 7 is devoted to the study of this important particular case after including a factor 2 in order to get the unit trace operator

$$(4.6) \quad \mathbf{M}^{a\mathcal{W}} \equiv 2\mathcal{I} = \int_{\Pi_+} U(q, p) \varpi_{a\mathcal{W}}(q, p) dq dp, \quad \varpi_{a\mathcal{W}}(q, p) := \frac{e^{-i\sqrt{q}p}}{\sqrt{q}}.$$

This operator is the affine counterpart of the operator yielding the Weyl-Wigner integral quantization when the phase space is \mathbb{R}^2 , i.e. we deal with Weyl-Heisenberg symmetry. Its unit trace property is proved below.

Let us now establish a necessary condition on ϖ to have a unit trace operator \mathbf{M}^ϖ .

Proposition 4.1. *Suppose that the operator \mathbf{M}^ϖ is unit trace class $\text{Tr}(\mathbf{M}^\varpi) = 1$. Then its corresponding weight function $\varpi(q, p)$ obeys*

$$(4.7) \quad \frac{\varpi(1, 0)}{2} + \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}/[-\epsilon, \epsilon]} \frac{\varpi(1, p)}{p} dp = 1.$$

Proof. From the definition (4.1) of the operator \mathbf{M}^ϖ the first step of the proof consists in determining the trace of the operator $\mathbf{C}_{\text{DM}}^{-1} U(q, p) \mathbf{C}_{\text{DM}}^{-1}$. For that we use the resolution of the identity (3.12) provided by an admissible vector ψ . We choose here a real ψ for convenience. We have successively

$$\begin{aligned} \text{Tr}(\mathbf{C}_{\text{DM}}^{-1} U(q, p) \mathbf{C}_{\text{DM}}^{-1}) &= \frac{1}{\|\mathbf{C}_{\text{DM}}\psi\|^2} \int_{\Pi_+} dq' dp' \times \\ &\quad \times \langle \psi | U^\dagger(q', p') \mathbf{C}_{\text{DM}}^{-1} U(q, p) \mathbf{C}_{\text{DM}}^{-1} U(q', p') | \psi \rangle \\ &= \frac{1}{2\pi q \|\mathbf{C}_{\text{DM}}\psi\|^2} \int_0^{+\infty} \frac{dq'}{q'} \int_0^{+\infty} dx x e^{ipx} \psi\left(\frac{x}{q'}\right) \psi\left(\frac{x}{qq'}\right) \times \\ &\quad \times \int_{-\infty}^{+\infty} dp' e^{-ip'x\left(\frac{q-1}{q}\right)}. \end{aligned}$$

Integrating on p' gives $2\pi\delta\left(x\frac{q-1}{q}\right) = 2\pi\frac{q^2}{x}\delta(q-1)$, which allows to fix $q = 1$. Then the change of variable $q' \mapsto y = x/q'$ allows to separate the two remaining integrals. That one on $(\psi(y))^2/y$ yields $\|\mathbf{C}_{\text{DM}}\psi\|^2/(2\pi)$ and the last one is the inverse Fourier transform of the (Heaviside) step function which is equal to $\frac{1}{\sqrt{2\pi}}\left(\pi\delta(p) + \text{i p.v.}\frac{1}{p}\right)$,

where p.v. denotes the Cauchy principal value. Finally,

$$(4.8) \quad \text{Tr} \left(\mathbf{C}_{\text{DM}}^{-1} U(q, p) \mathbf{C}_{\text{DM}}^{-1} \right) = \frac{1}{2} \delta(q-1) \left(\delta(p) + \frac{i}{\pi} \text{p.v.} \frac{1}{p} \right).$$

□

Note that the condition (4.7) can be also written as

$$(4.9) \quad \text{Tr}(\mathbf{M}^\varpi) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} dx \, \hat{\varpi}_p(1, -x) = 1,$$

where $\hat{\varpi}_p$ is the partial Fourier transform of ϖ with respect to the variable p defined as

$$(4.10) \quad \hat{\varpi}_p(q, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ipx} \varpi(q, p) \, dp.$$

From the Dirichlet integral $\int_0^{+\infty} \text{sinc} t \, dt = \frac{\pi}{2}$, we easily check that the special weight $\varpi_{a\mathcal{W}}(q, p)$ given in Eq. (4.6) satisfies the condition (4.7).

Suitably weighted and scaled diagonal matrix elements $U_{mm}^{(\alpha)}(q, p)$ are example of solutions to the functional equation (4.2). Indeed, let us define

$$(4.11) \quad \varpi_m^{(\alpha)}(q, p) := \frac{1}{\sqrt{q}} U_{mm}^{(\alpha)}(q, p) = \frac{2^{\alpha+1} q^{\alpha/2}}{(q+1+2iqp)^{\alpha+1}} e^{2im\theta(q,p)} P_m^{(0,\alpha)}(Y(q,p)),$$

where $\theta(q, p) = \arg(q+1+2iqp)$. From (3.8), one easily checks that the s -dependent weight function defined by

$$(4.12) \quad \varpi_{m;s}^{(\alpha)}(q, p) := \varpi_m^{(\alpha)}(q, sp),$$

verifies (4.2) for any $s \in \mathbb{R}$ and $m \in \mathbb{N}$. With this function in hand we have the following interesting result.

Proposition 4.2. *Let $\mathbf{P}_m = |e_m^{(\alpha)}\rangle\langle e_m^{(\alpha)}|$ be the orthogonal projector on the Laguerre basis element $e_m^{(\alpha)}(x)$ and let $\rho_{m;s}^{(\alpha)}$ be the self-adjoint operator defined by*

$$(4.13) \quad \rho_{m;s}^{(\alpha)} = \int_{\Pi_+} U(q, p) \varpi_{m;s}^{(\alpha)}(q, p) \, dq \, dp, \quad \text{with } s > 0.$$

Then we have

$$(4.14) \quad \sqrt{Q} U^\dagger(s, 0) \rho_{m;s}^{(\alpha)} U(s, 0) \sqrt{Q} = 2\pi \mathbf{P}_m,$$

or equivalently

$$(4.15) \quad \mathbf{P}_m^{(\alpha)} = \int_{\Pi_+} \mathbf{C}_{\text{DM}}^{-1} U(q, sp) \mathbf{C}_{\text{DM}}^{-1} \varpi_{m;s}^{(\alpha)}(q, p) \, dq \, dp.$$

Proof. Let us first apply (4.13) to $\psi \in L^2(\mathbb{R}_+, dx)$ while keeping the integral form (3.5) of $U_{mm}^{(\alpha)}$.

$$\begin{aligned}
 (\rho_{m;s}^{(\alpha)} \psi)(x) &= \int_{\Pi_+} dq dp U(q, p) \psi(x) \frac{1}{\sqrt{q}} U_{mm}^{(\alpha)}(q, sp) \\
 &= \frac{m!}{(m+\alpha)!} \int_{\Pi_+} \frac{dq dp}{q^{\frac{\alpha+3}{2}}} \int_0^\infty dy y^\alpha e^{-\left(\frac{1}{2} + \frac{1}{2q} - isp\right)y + ipx} \times \\
 (4.16) \quad &\times L_m^{(\alpha)}(y) L_m^{(\alpha)}\left(\frac{y}{q}\right) \psi\left(\frac{x}{q}\right).
 \end{aligned}$$

After performing the integration on p , which gives the Dirac $2\pi\delta(x - sy)$, and then integrating on y , one obtains

$$(4.17) \quad (\rho_{m;s}^{(\alpha)} \psi)(x) = \frac{2\pi m!}{(m+\alpha)!} e^{-\frac{x}{2s}} \left(\frac{x}{s}\right)^\alpha L_m^{(\alpha)}\left(\frac{x}{s}\right) \int_0^\infty \frac{dq}{q^{\frac{\alpha+3}{2}}} e^{-\frac{x}{2sq}} L_m^{(\alpha)}\left(\frac{x}{sq}\right) \psi\left(\frac{y}{q}\right).$$

After performing the change of variable $q \rightarrow x/q = u$, we obtain

$$\begin{aligned}
 (\rho_{m;s}^{(\alpha)} \psi)(x) &= \frac{2\pi m!}{(m+\alpha)!} e^{-\frac{x}{2s}} \left(\frac{x}{s}\right)^{\frac{\alpha-1}{2}} L_m^{(\alpha)}\left(\frac{x}{s}\right) \times \\
 &\times \int_0^\infty \frac{du}{s} e^{-\frac{u}{2s}} \left(\frac{u}{s}\right)^{\frac{\alpha-1}{2}} L_m^{(\alpha)}\left(\frac{u}{s}\right) \psi(u) \\
 (4.18) \quad &= 2\pi \left(U(s, 0) \sqrt{\frac{1}{Q}} e_m^{(\alpha)} \right)(x) \left\langle U(s, 0) \sqrt{\frac{1}{Q}} e_m^{(\alpha)} \middle| \psi \right\rangle.
 \end{aligned}$$

□

This result is interesting since it allows to give the thermal state ρ_t in (3.15) the integral representation

$$\begin{aligned}
 \rho_t^{(\alpha)} &= \frac{1-t}{2\pi} \sum_{n=0}^\infty t^n \sqrt{Q} \rho_{n;1} \sqrt{Q} = \frac{1-t}{2\pi} \int_{\Pi_+} dq dp \sqrt{Q} U(q, p) \sqrt{Q} \frac{1}{\sqrt{q}} \sum_{n=0}^\infty t^n U_{nn}^{(\alpha)}(q, p) \\
 (4.19) \quad &= \int_{\Pi_+} dq dp \sqrt{Q} U(q, p) \sqrt{Q} \varpi_t^{(\alpha)}(q, p),
 \end{aligned}$$

where the weight $\varpi_t^{(\alpha)}(q, p)$ has an involved form issue from a generating function of Jacobi polynomials found in page 213 of [27],

$$(4.20) \quad \varpi_t^{(\alpha)}(q, p) = \frac{1-t}{2\pi} \frac{2^{2\alpha+1} q^{\frac{\alpha}{2}}}{\mathcal{R}(q, p; t)} \left(\frac{\bar{Z}_+}{Z_+} \right)^{\alpha+1} (\bar{Z}_+ + tZ_+ + \mathcal{R}(q, p; t))^{-\alpha},$$

with

$$(4.21) \quad \mathcal{R}(q, p; t) = (\bar{Z}_+^2 - 2Yt|Z_+|^2 + t^2 Z_+^2)^{1/2}.$$

Finally note the integral representation of the simplest case $t = 0$ which corresponds to the projector on the first basis element,

$$(4.22) \quad \rho_0^{(\alpha)} = P_0^{(\alpha)} = |e_0^{(\alpha)}\rangle\langle e_0^{(\alpha)}| = \frac{2^{\alpha+1}}{2\pi} \int_{\Pi_+} dq dp \frac{q^{\alpha/2}}{(q+1+2iqp)^{\alpha+1}} \sqrt{Q} U(q, p) \sqrt{Q}.$$

This operator is defined through its action in $L^2(\mathbb{R}_+^*, dx)$ by

$$(4.23) \quad (\rho_0^{(\alpha)} \psi)(x) = \frac{1}{\alpha!} x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} \int_0^{+\infty} dy y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} \psi(y).$$

5. COVARIANT AFFINE INTEGRAL QUANTIZATION FROM WEIGHT

5.1. General results. We now start from the framework of Section 4 and establish general formulas for quantization issued from a weight function $\varpi(q, p)$, which obeys Eq. (4.2) and so yields a symmetric operator as

$$(5.1) \quad \int_{\Pi_+} C_{DM}^{-1} U(q, p) C_{DM}^{-1} \varpi(q, p) dq dp := M^\varpi.$$

Let us first establish the nature of M^ϖ as an integral operator in $\mathcal{H} = L^2(\mathbb{R}_+^*, dx)$.

Proposition 5.1. *The action on ϕ in \mathcal{H} of the operator M^ϖ defined by the integral representation (5.1) is given by*

$$(5.2) \quad (M^\varpi \phi)(x) = \int_0^{+\infty} \mathcal{M}^\varpi(x, x') \phi(x') dx',$$

where the kernel \mathcal{M}^ϖ is given by

$$(5.3) \quad \mathcal{M}^\varpi(x, x') = \frac{1}{\sqrt{2\pi}} \frac{x}{x'} \hat{\varpi}_p \left(\frac{x}{x'}, -x \right).$$

Here $\hat{\varpi}_p$ is the partial Fourier transform of ϖ with respect to the variable p , as it was defined by Eq. (4.10).

Proof. Let ϕ_1, ϕ_2 be two elements of \mathcal{H} . Supposing that the expression $\langle \phi_1 | M^\varpi | \phi_2 \rangle$ is finite, we have from the action on the right of $U(q, p)$ and of the Duflo-Moore operators

$$\begin{aligned} \langle \phi_1 | M^\varpi | \phi_2 \rangle &= \frac{1}{2\pi} \int_0^{+\infty} \frac{dq}{q} \int_0^{+\infty} dx x \overline{\phi_1(x)} \phi_2 \left(\frac{x}{q} \right) \int_{-\infty}^{+\infty} dp e^{ipx} \varpi(q, p) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \frac{dq}{q} \int_0^{+\infty} dx x \overline{\phi_1(x)} \phi_2 \left(\frac{x}{q} \right) \hat{\varpi}_p(q, -x) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} dx \int_0^{+\infty} dx' \overline{\phi_1(x)} \phi_2(x') \frac{x}{x'} \hat{\varpi}_p \left(\frac{x}{x'}, -x \right), \end{aligned}$$

the last equation being obtained through the change of variables $q \mapsto x' = x/q$. \square

We note that, if we impose M^ϖ to be symmetric operator, the resulting symmetry of the kernel

$$(5.4) \quad \mathcal{M}^\varpi(x, x') = \overline{\mathcal{M}^\varpi(x', x)}$$

is trivially derived from Eq. (4.2) and basic properties of the Fourier transform.

Corollary 5.1. *Let the operator \mathbf{M}^ϖ be a pure state $|\psi\rangle\langle\psi|$ as it is for the construction of affine coherent states. Then the corresponding weight function is given through its partial Fourier transform by*

$$(5.5) \quad \hat{\omega}_p(u, v) = \sqrt{2\pi} \frac{1}{u} \psi(-v) \overline{\psi\left(-\frac{v}{u}\right)}, \quad u > 0, v < 0.$$

In particular, we have the relation for the modulus of ψ

$$(5.6) \quad \hat{\omega}_p(1, -x) = \sqrt{2\pi} |\psi(x)|^2.$$

Proof. Immediate from

$$\mathcal{M}^\varpi(x, x') = \psi(x) \overline{\psi(x')}.$$

□

Let us now carry out the integral quantization (2.5) with $G = \text{Aff}_+(\mathbb{R})$ and $\mathbf{M} = \mathbf{M}^\varpi$:

$$(5.7) \quad f \mapsto A_f^\varpi = \int_{\Pi_+} \frac{dq dp}{c_{\mathbf{M}^\varpi}} f(q, p) \mathbf{M}^\varpi(q, p).$$

By construction, the quantization map (5.7) is covariant with respect to the unitary affine action U :

$$(5.8) \quad U(q_0, p_0) A_f^\varpi U^\dagger(q_0, p_0) = A_{\varpi_{\mathfrak{U}(q_0, p_0)} f},$$

with

$$(5.9) \quad (\mathfrak{U}(q_0, p_0) f)(q, p) = f((q_0, p_0)^{-1}(q, p)) = f\left(\frac{q}{q_0}, q_0(p - p_0)\right),$$

\mathfrak{U} being the left regular representation of the affine group.

Proposition 5.2. *The action on ϕ in \mathcal{H} of the operator A_f^ϖ defined by the integral quantization map (5.7) is given by*

$$(5.10) \quad (A_f^\varpi \phi)(x) = \int_0^{+\infty} \mathcal{A}_f^\varpi(x, x') \phi(x') dx',$$

where the kernel \mathcal{A}_f^ϖ is defined as

$$(5.11) \quad \mathcal{A}_f^\varpi(x, x') = \frac{\sqrt{2\pi}}{c_{\mathbf{M}^\varpi}} \int_0^{+\infty} \frac{dq}{q} \mathcal{M}^\varpi\left(\frac{x}{q}, \frac{x'}{q}\right) \hat{f}_p(q, x' - x)$$

$$(5.12) \quad = \frac{1}{c_{\mathbf{M}^\varpi}} \frac{x}{x'} \int_0^{+\infty} \frac{dq}{q} \hat{\omega}_p\left(\frac{x}{x'}, -q\right) \hat{f}_p\left(\frac{x}{q}, x' - x\right).$$

Here \hat{f}_p is the partial Fourier transform of f with respect to the variable p defined in Eq. (4.10).

Proof. Let ϕ_1, ϕ_2 be two elements of \mathcal{H} . Supposing that the expression $\langle \phi_1 | A_f^\varpi | \phi_2 \rangle$ makes sense, we have from the action of $U^\dagger(q, p)$ on the right and of $U(q, p)$ on the left,

$$\begin{aligned} \langle \phi_1 | A_f^\varpi | \phi_2 \rangle &= \frac{1}{c_{\mathcal{M}^\varpi}} \int_0^{+\infty} dq q \int_0^{+\infty} dx \int_0^{+\infty} dx' \overline{\phi_1(qx)} \mathcal{M}^\varpi(x, x') \phi_2(qx') \times \\ &\quad \times \int_{-\infty}^{+\infty} dp e^{-iqp(x'-x)} f(q, p) \\ &= \frac{\sqrt{2\pi}}{c_{\mathcal{M}^\varpi}} \int_0^{+\infty} dq q \int_0^{+\infty} dx \int_0^{+\infty} dx' \overline{\phi_1(qx)} \times \\ &\quad \times \mathcal{M}^\varpi(x, x') \phi_2(qx') \hat{f}_p(q, q(x' - x)) \\ &= \frac{\sqrt{2\pi}}{c_{\mathcal{M}^\varpi}} \int_0^{+\infty} dx \int_0^{+\infty} dx' \overline{\phi_1(x)} \phi_2(x') \int_0^{+\infty} \frac{dq}{q} \mathcal{M}^\varpi\left(\frac{x}{q}, \frac{x'}{q}\right) \hat{f}_p(q, x' - x), \end{aligned}$$

the last equation being obtained through the changes of variables $qx \mapsto x$ and $qx' \mapsto x'$. Eq. (5.12) is obtained by using (5.3) and the change $q \mapsto x/q$. \square

From the general form (5.12) of the integral kernel and particularizing to $f = 1$, i.e. $\hat{f}_p\left(\frac{x}{q}, x' - x\right) = \sqrt{2\pi} \delta(x' - x)$, case which corresponds to the resolution of the identity, we get the relation between the normalisation constant $c_{\mathcal{M}^\varpi}$ and an integral on the weight function.

Corollary 5.2. *A necessary condition for having the resolution of the identity issued from a choice of weight function $\varpi(q, p)$ is*

$$(5.13) \quad c_{\mathcal{M}^\varpi} = \sqrt{2\pi} \int_0^{+\infty} \frac{dq}{q} \hat{\varpi}_p(1, -q) \equiv \sqrt{2\pi} \Omega(1) < \infty,$$

where we have introduced the function

$$(5.14) \quad \Omega(u) = \int_0^{+\infty} \frac{dq}{q} \hat{\varpi}_p(u, -q).$$

5.2. Particular cases.

Position dependent functions f . Suppose that f does not depend on p , $f(q, p) \equiv u(q)$. From

$$\hat{f}_p(q, x' - x) = \sqrt{2\pi} u(q) \delta(x' - x),$$

and after integration one obtains for (5.11)

$$\begin{aligned} \mathcal{A}_u^\varpi(x, x') &= \frac{2\pi}{c_{\mathcal{M}^\varpi}} \delta(x - x') \int_0^{+\infty} \frac{dq}{q} \mathcal{M}^\varpi\left(\frac{x}{q}, \frac{x}{q}\right) u(q) \\ (5.15) \quad &= \frac{2\pi}{c_{\mathcal{M}^\varpi}} \delta(x - x') \int_0^{+\infty} \frac{dq}{q} \mathcal{M}^\varpi(q, q) u\left(\frac{x}{q}\right). \end{aligned}$$

Thus, the quantum version of the function $u(q)$ is the multiplication operator

$$(5.16) \quad A_{u(q)}^\varpi = \frac{2\pi}{c_{\mathcal{M}^\varpi}} \int_0^{+\infty} \frac{dq}{q} \mathcal{M}^\varpi(q, q) u\left(\frac{Q}{q}\right) = \frac{\sqrt{2\pi}}{c_{\mathcal{M}^\varpi}} \int_0^{+\infty} \frac{dq}{q} \hat{\varpi}_p(1, -q) u\left(\frac{Q}{q}\right)$$

i.e. the multiplication by the convolution on the multiplicative group \mathbb{R}_+^* of $u(x)$ with $\frac{\sqrt{2\pi}}{c_{\mathcal{M}^\varpi}} \hat{\omega}_p(1, -x)$.

An interesting more particular case is when u is a simple power of q , say $u(q) = q^\beta$. Then we have

$$(5.17) \quad A_{q^\beta}^\varpi = \frac{\sqrt{2\pi}}{c_{\mathcal{M}^\varpi}} \int_0^{+\infty} \frac{dq}{q^{1+\beta}} \hat{\omega}_p(1, -q) Q^\beta \equiv \frac{d_\beta}{d_0} Q^\beta,$$

where we have introduced the convenient notation

$$(5.18) \quad d_\beta = \int_0^{+\infty} \frac{dq}{q^{1+\beta}} \hat{\omega}_p(1, -q),$$

together with necessary conditions of convergence. Note that with this notation,

$$(5.19) \quad c_{\mathcal{M}^\varpi} = \sqrt{2\pi} d_0.$$

Momentum dependent functions f . Now suppose that f does not depend on q , $f(q, p) \equiv v(p)$. The formula (5.12) simplifies to

$$(5.20) \quad \mathcal{A}_{v(p)}^\varpi(x, x') = \frac{1}{c_{\mathcal{M}^\varpi}} \hat{v}(x' - x) \frac{x}{x'} \int_0^{+\infty} \frac{dq}{q} \hat{\omega}_p\left(\frac{x}{x'}, -q\right) = \frac{1}{c_{\mathcal{M}^\varpi}} \hat{v}(x' - x) \frac{x}{x'} \Omega\left(\frac{x}{x'}\right).$$

As a simple but important example, let us examine the case $v(p) = p^n$, $n \in \mathbb{N}$. From distribution theory

$$(5.21) \quad \hat{v}(x' - x) = \sqrt{2\pi} i^n \delta^{(n)}(x' - x),$$

we derive the differential action of the operator $A_{p^n}^\varpi$ in \mathcal{H} as the polynomial in $P = -i d/dx$

$$(5.22) \quad A_{p^n}^\varpi = \frac{\sqrt{2\pi}}{c_{\mathcal{M}^\varpi}} \sum_{k=0}^n \binom{n}{k} \left(-i \frac{d}{dx'}\right)^{n-k} \frac{x}{x'} \Omega\left(\frac{x}{x'}\right) \Big|_{x'=x} P^k = P^n + \dots.$$

In particular

$$(5.23) \quad A_p^\varpi = P + \frac{i}{x} \left[1 + \frac{\Omega'(1)}{\Omega(1)} \right].$$

This operator is symmetric but has no self-adjoint extension [29]. Hence, from (5.17) with $\beta = 1$, the canonical commutation rule holds up to a factor which can be easily put equal to one through a rescaling of the weight function

$$(5.24) \quad [A_q, A_p] = \frac{d_\beta}{d_0} i I.$$

For the kinetic energy we have

$$(5.25) \quad A_{p^2}^\varpi = P^2 + \frac{2i}{Q} \left[1 + \frac{\Omega'(1)}{\Omega(1)} \right] P - \frac{1}{Q^2} \left[2 + 4 \frac{\Omega'(1)}{\Omega(1)} + \frac{\Omega''(1)}{\Omega(1)} \right].$$

This symmetric operator is essentially self-adjoint or not, depending on the strength of the (attractive or repulsive) potential $1/x^2$ [29]. With the choice of a weight function such that $-2 - 4 \frac{\Omega'(1)}{\Omega(1)} - \frac{\Omega''(1)}{\Omega(1)} \geq 3/4$, it is essentially self-adjoint and so quantum dynamics of the free motion on the half line is unique.

Separable functions f . Finally, suppose that f is separable, i.e. $f(q, p) \equiv u(q) v(p)$. The formula (5.12) simplifies to

$$(5.26) \quad \mathcal{A}_{u(q)v(p)}^\varpi(x, x') = \frac{1}{c_{\mathbf{M}^\varpi}} \hat{v}(x' - x) \frac{x}{x'} \int_0^{+\infty} \frac{dq}{q} \hat{\varpi}_p\left(\frac{x}{x'}, -q\right) u\left(\frac{x}{q}\right).$$

The elementary example is the quantization of the function qp which produces the integral kernel and its corresponding operator

$$(5.27) \quad \begin{aligned} \mathcal{A}_{qp}^\varpi(x, x') &= \frac{\sqrt{2\pi}}{c_{\mathbf{M}^\varpi}} i \delta'(x' - x) \frac{x^2}{x'} \int_0^{+\infty} \frac{dq}{q^2} \hat{\varpi}_p\left(\frac{x}{x'}, -q\right), \\ A_{qp}^\varpi &= \frac{\Omega_1(1)}{\Omega(1)} D + i \left[\frac{3}{2} \frac{\Omega_1(1)}{\Omega(1)} + \frac{\Omega'_1(1)}{\Omega(1)} \right], \end{aligned}$$

where $D = \frac{1}{2}(QP + PQ)$ is the dilation generator. As one of the two generators (with Q) of the UIR U of the affine group, it is essentially self-adjoint, with continuous spectrum $\lambda \in \mathbb{R}$ and corresponding eigendistributions $x^{\frac{1}{2}+i\lambda}$. We have introduced in (5.27) one more notation with

$$(5.28) \quad \Omega_\beta(u) = \int_0^{+\infty} \frac{dq}{q^{1+\beta}} \hat{\varpi}_p(u, -q), \quad \Omega_0(u) = \Omega(u).$$

5.3. Semi-classical portraits. Given a weight function $\varpi(q, p)$ yielding a symmetric unit trace operator \mathbf{M}^ϖ , we define the semi-classical or lower symbol of an operator A in \mathcal{H} as the function

$$(5.29) \quad \check{A}(q, p) := \text{Tr} \left(A U(q, p) \mathbf{M}^\varpi U^\dagger(q, p) \right) = \text{Tr} (A \mathbf{M}^\varpi(q, p)).$$

When the operator A is the affine integral quantized version of a classical $f(q, p)$ with the same weight ϖ , we get the transform of the type (2.7)

$$(5.30) \quad f(q, p) \mapsto \check{f}(q, p) \equiv \check{A}^\varpi(q, p) = \int_{\Pi_+} \frac{dq' dp'}{c_{\mathbf{M}^\varpi}} f\left(qq', \frac{p'}{q} + p\right) \text{Tr} (\mathbf{M}^\varpi(q', p') \mathbf{M}^\varpi).$$

Of course, this expression has the meaning of an averaging of the classical f if the function

$$(5.31) \quad \begin{aligned} (q, p) \equiv g &\mapsto \frac{1}{c_{\mathbf{M}^\varpi}} \text{Tr} (\mathbf{M}^\varpi(g) \mathbf{M}^\varpi) = \\ &= \frac{1}{c_{\mathbf{M}^\varpi}} \int_{\Pi_+} dg_1 \varpi(g_1) \int_{\Pi_+} dg_2 \varpi(g_2) \times \\ &\times \text{Tr} (U(g) \mathbf{C}_{\text{DM}}^{-1} U(g_1) \mathbf{C}_{\text{DM}}^{-1} U(g^{-1}) \mathbf{C}_{\text{DM}}^{-1} U(g_2) \mathbf{C}_{\text{DM}}^{-1}) \end{aligned}$$

is a true probability distribution on the half-plane, i.e. is positive since we know from the resolution of the identity that its integral is 1. So a new trace formula, extending (4.8), is needed here. Explicitly,

$$(5.32) \quad \text{Tr} (\mathbf{M}^\varpi(q, p) \mathbf{M}^\varpi) = \frac{1}{2\pi q} \int_0^{+\infty} dx \int_0^{+\infty} dy e^{-ip(y-x)} \hat{\varpi}_p\left(\frac{x}{y}, -\frac{x}{q}\right) \hat{\varpi}_p\left(\frac{y}{x}, -y\right).$$

Integrating this expression on Π_+ with the measure $\frac{dq dp}{c_{\mathbf{M}^\varpi}}$ and using (4.9) (i.e., $d_1 = \text{Tr}(\mathbf{M}^\varpi) = 1$) and (5.13), we get 1, which means that $\hat{1} = 1$, as expected.

6. QUANTIZATION WITH AFFINE CS AND THERMAL STATES

6.1. Quantization with ACS. Let us first implement the integral quantization scheme described above by restricting the method to the specific case of rank-one density operator or projector $\mathbf{M}^\varpi = |\psi\rangle\langle\psi|$ where ψ is a unit-norm admissible state, i.e. is in $L^2(\mathbb{R}_+^\dagger, dx) \cap L^2(\mathbb{R}_+^\dagger, dx/x)$ (such a ψ is also called “fiducial vector” or “wavelet”). With the notations of the previous section, we have from (5.5) and (5.6)

$$(6.1) \quad \Omega(u) = \frac{\sqrt{2\pi}}{u} \int_0^{+\infty} \frac{dq}{q} \psi(q) \overline{\psi\left(\frac{q}{u}\right)}, \quad d_\beta = \sqrt{2\pi} \int_0^{+\infty} \frac{dq}{q^{1+\beta}} |\psi(q)|^2.$$

In particular,

$$(6.2) \quad \Omega(1) = d_0, \quad \Omega'(1) = -\Omega(1) - \sqrt{2\pi} \langle \psi' | \psi \rangle,$$

$$(6.3) \quad \Omega''(1) = 2\Omega(1) + 4\sqrt{2\pi} \langle \psi' | \psi \rangle + \sqrt{2\pi} \langle \psi'' | Q | \psi \rangle.$$

Note that $\langle \psi' | \psi \rangle$ is purely imaginary and cancels for real ψ .

Therefore, by applying the general formalism, we recover a set of results already given in previous works, e.g. in [5]. As was already pointed out after stating the orthogonality relations (3.9), the action of the UIR operators $U(q, p)$ on ψ produces all affine coherent states, i.e. wavelets, defined as $|q, p\rangle = U(q, p)|\psi\rangle$. Immediate examples of such vectors are $\psi(x) = e_m^{(\alpha)}(x)$, i.e., with $\rho_m = \mathbf{P}_m$, for $\alpha \geq \alpha_0 > 0$, where α_0 is suitably chosen in view of quantizing a certain class of function $f(q, p)$.

Hence, to the irreducibility of the representation U and its square-integrability expressed by (3.9), the corresponding quantization reads as

$$(6.4) \quad f \mapsto A_f = \int_{\Pi_+} f(q, p) |q, p\rangle\langle q, p| \frac{dq dp}{2\pi c_{-1}},$$

which arises from the resolution of the identity

$$(6.5) \quad \int_{\Pi_+} |q, p\rangle\langle q, p| \frac{dq dp}{2\pi c_{-1}} = I,$$

where we adopt for convenience the notations of [5],

$$(6.6) \quad c_\gamma := \int_0^\infty |\psi(x)|^2 \frac{dx}{x^{2+\gamma}} = \frac{1}{\sqrt{2\pi}} d_{\gamma+1}.$$

Thus, a necessary condition to have (6.5) true is that $c_{-1} < \infty$, which implies $\psi(0) = 0$, a well-known requirement in wavelet analysis.

To simplify, we choose a real fiducial vector. Then,

$$(6.7) \quad A_p = P, \quad A_{q^\beta} = \frac{c_{\beta-1}}{c_{-1}} Q^\beta.$$

Whereas Q is self-adjoint, we recall that the operator P is symmetric but has no self-adjoint extension. The quantization of the product qp yields:

$$(6.8) \quad A_{qp} = \frac{c_0}{c_{-1}} \frac{QP + PQ}{2} = \frac{c_0}{c_{-1}} D.$$

The quantization of kinetic energy gives

$$(6.9) \quad A_{p^2} = P^2 + KQ^{-2}, \quad K = K(\psi) = \int_0^\infty (\psi'(u))^2 u \frac{du}{c_{-1}}.$$

Therefore, wavelet quantization prevents a quantum free particle moving on the positive line from reaching the origin. As already discussed in the previous section, the above regularized operator, defined on the domain of smooth function of compact support, is essentially self-adjoint for $K \geq 3/4$ [30], and then quantum dynamics of the free motion on the half line is unique. Whilst canonical quantization, based on Weyl-Heisenberg symmetry which is unnatural in the present case, introduces ambiguity on the quantum level, ACS quantization with suitable fiducial vector removes this ambiguity.

The quantum states and their dynamics have semi-classical phase space representations through symbols. For the state $|\phi\rangle$ the corresponding symbol reads

$$(6.10) \quad \Phi(q, p) = \langle q, p | \phi \rangle / \sqrt{2\pi},$$

with the associated probability distribution on phase space given by

$$(6.11) \quad \rho_\phi(q, p) = \frac{1}{2\pi c_{-1}} |\langle q, p | \phi \rangle|^2.$$

Having the (energy) eigenstates of some quantum Hamiltonian \mathbf{H} at our disposal, the most natural being in this context the quantized A_h of a classical Hamiltonian $h(q, p)$, we can compute the time evolution

$$(6.12) \quad \rho_\phi(q, p, t) := \frac{1}{2\pi c_{-1}} |\langle q, p | e^{-i\mathbf{H}t} | \phi \rangle|^2$$

for any state ϕ .

The map (2.7) yielding lower symbols from classical f reads in the present case (supposing that Fubini holds):

$$(6.13) \quad \check{f}(q, p) = \frac{1}{\sqrt{2\pi c_{-1}}} \int_0^\infty \frac{dq'}{qq'} \int_0^\infty dx \int_0^\infty dx' e^{ip(x'-x)} \times \\ \times \hat{f}_p(q', x' - x) \psi\left(\frac{x}{q}\right) \psi\left(\frac{x}{q'}\right) \psi\left(\frac{x'}{q}\right) \psi\left(\frac{x'}{q'}\right),$$

where \hat{f}_p stands for the partial inverse Fourier transform introduced in (4.10).

For functions f depending on q only, expression (6.13) simplifies to a lower symbol depending on q only:

$$(6.14) \quad \check{f}(q) = \frac{1}{c_{-1}} \int_0^\infty \frac{dq'}{qq'} f(q') \int_0^\infty dx \psi^2\left(\frac{x}{q}\right) \psi^2\left(\frac{x}{q'}\right).$$

For instance, any power of q is transformed into the same power up to a constant factor

$$(6.15) \quad q^\beta \mapsto \check{q}^\beta = \frac{c_{-\beta-1}c_{-\beta-2}}{c_{-1}} q^\beta.$$

Note that $c_{-2} = 1$ from the normalisation of ψ . Other important symbols are:

$$(6.16) \quad p \mapsto \check{p} = p,$$

$$(6.17) \quad p^2 \mapsto \check{p}^2 = p^2 + \frac{c(\psi)}{q^2}, \quad c(\psi) = \int_0^\infty (\psi'(x))^2 (1 + c_1 x) dx.$$

$$(6.18) \quad qp \mapsto \check{q}\check{p} = \frac{c_0 c_{-3}}{c_{-1}} qp$$

Another interesting formula in the semi-classical context concerns the Fubini-Study metric derived from the symbol of total differential d with respect to parameters q and p affine coherent states,

$$(6.19) \quad \langle q, p | d | q, p \rangle = iq dp \int_0^\infty (\psi(x))^2 x dx = iq dp c_{-3}.$$

and from norm squared of $d|q, p\rangle$,

$$(6.20) \quad \|d|q, p\rangle\|^2 = c_{-4} q^2 dp^2 + L \frac{dq^2}{q^2}, \quad L = \int_0^\infty dx x^2 (\psi'(x))^2 - \frac{1}{4}.$$

With Klauder's notations [31]

$$(6.21) \quad d\sigma^2(q, p) := 2 [\|d|q, p\rangle\|^2 - |\langle q, p | d | q, p \rangle|^2] = 2 \left((c_{-4} - c_{-3}^2) q^2 dp^2 + L \frac{dq^2}{q^2} \right).$$

6.2. Quantization of basic observables with Laguerre thermal state. In this subsection we compute the quantized version of $f(q, p)$, e.g. the momentum p , the classical dilation qp , the kinetic energy p^2 , the power potential q^β when the affine integral quantization is carried out with the thermal density operator $\rho_t(q, p)$. Because the closed formula (4.20) for the weight function $\varpi_{\text{th}}(q, p; t)$ is quite intricate, it is more tractable to work directly with the expansion (3.15) of ρ_t , to use the ACS quantization formulae above for each rank one operator in the series and to sum the results. With the notations (3.23), (4.11), and (4.19), the general formula reads as

$$(6.22) \quad A_f^{\varpi_t^{(\alpha)}} = \frac{\alpha}{2\pi} \int_{\Pi_+} \rho_t^{(\alpha)}(q, p) f(q, p) dq dp = \frac{\alpha(1-t)}{2\pi} \sum_{n=0}^\infty t^n c_{-1;n}^{(\alpha)} A_f^{\varpi_n^{(\alpha)}},$$

where the constants

$$(6.23) \quad c_{\gamma;n}^{(\alpha)} := \frac{n!}{(n+\alpha)!} \int_0^{+\infty} \frac{dx}{x^{2+\gamma}} e^{-x} x^\alpha \left(L_n^{(\alpha)}(x) \right)^2$$

can easily be deduced from 7.414 Eq. (12) in [28].

Proposition 6.1. *The affine integral quantizations with Laguerre thermal state of the classical momentum, the kinetic energy, the dilation function qp , and the powers of q are given by*

$$(6.24) \quad A_p^{\varpi_t^{(\alpha)}} = P.$$

$$(6.25) \quad A_{q^\beta}^{\varpi_t^{(\alpha)}} = c_{\beta-1}^{(\alpha)}(t) Q^\beta, \quad c_\gamma^{(\alpha)}(t) := \frac{\alpha(1-t)}{2\pi} \sum_{n=0}^{\infty} t^n c_{\beta-1;n}^{(\alpha)}.$$

$$(6.26) \quad A_{qp}^{\varpi_t^{(\alpha)}} = c_0^{(\alpha)}(t) D.$$

$$(6.27) \quad A_{p^2}^{\varpi_t^{(\alpha)}} = P^2 + \frac{K^{(\alpha)}(t)}{Q^2}, \quad K^{(\alpha)}(t) := \frac{\alpha(1-t)}{2\pi} \sum_{n=0}^{\infty} t^n \int_0^\infty \left(\frac{de_n^{(\alpha)}(x)}{dx} \right)^2 x dx.$$

Proof. These formulas are proved by the fact that all fiducial Laguerre basis elements are real and direct application of Eqs. (6.7) (6.9) and (6.9). \square

7. THE INVERSE AFFINE OPERATOR AND QUANTIZATION OF BASIC OPERATORS

In this section we investigate the integral quantization yielded by $M^{a\mathcal{W}}$. This operator is equal to twice the inversion operator \mathcal{I} defined on $L^2(\mathbb{R}_+, dx)$ as it was introduced in Eq. (4.5).

7.1. Properties of the inversion map and the related quantization. First of all let us return to the determination of the trace of the inversion operator \mathcal{I} .

Proposition 7.1. *The operator $M^{a\mathcal{W}} = 2\mathcal{I}$ is unit trace.*

Proof. Although we have already proved this property by application of Proposition 4.1, it is useful to present here a direct proof through the use of the orthonormal Laguerre basis with $\alpha = 0$. One gets successively

$$\begin{aligned} \text{Tr}(\mathcal{I}) &= \sum_{n=0}^{\infty} \langle e_n^{(0)} | \mathcal{I} e_n^{(0)} \rangle = \sum_{n=0}^{\infty} \int_0^\infty \frac{dx}{x} e^{-\frac{x}{2}} e^{-\frac{1}{2x}} L_n(x) L_n\left(\frac{1}{x}\right) \\ &= \int_0^\infty \frac{dx}{x} \sum_{n=0}^{\infty} e^{-\frac{x}{2}} e^{-\frac{1}{2x}} L_n(x) L_n\left(\frac{1}{x}\right) = \int_0^\infty \frac{dx}{x} \delta\left(x - \frac{1}{x}\right) \\ &= \int_0^\infty \frac{dx}{x} \frac{\delta(x-1)}{2} = \frac{1}{2} \end{aligned}$$

\square

Proposition 7.2. *The integral kernel (5.12) of the quantization of a function $f(q, p)$ through the weight function $\varpi_{a\mathcal{W}}$ given in (4.6) has the following expression,*

$$(7.1) \quad \mathcal{A}_f^{a\mathcal{W}}(x, x') = \frac{1}{\sqrt{2\pi}} \hat{f}_p \left(\sqrt{\frac{x'}{x}}, x' - x \right).$$

Proof. The computation of the partial Fourier transform of the weight function $\varpi_{a\mathcal{W}}(q, p) = e^{-i\sqrt{q}p}/\sqrt{q}$ is immediate and yields

$$(7.2) \quad \widehat{(\varpi_{a\mathcal{W}})}_p(q, k) = \sqrt{2\pi} \frac{\delta(k + \sqrt{q})}{\sqrt{q}}.$$

It follows for the integral $\Omega(u)$ defined by (5.14) and the constant (5.13) the simple values

$$(7.3) \quad \Omega(u) = \frac{\sqrt{2\pi}}{u}, \quad c_{\varpi_{a\mathcal{W}}} = 2\pi.$$

Then, Eq. (7.1) results from the integration with delta distribution. \square

We derive from Proposition 7.2 the following interesting results holding for this particular type of integral quantization.

- Proposition 7.3.** (i) *The quantization of a function of q , $f(q, p) = u(q)$ provided by the weight $\varpi_{a\mathcal{W}}$ is $u(Q)$.*
(ii) *Similarly, the quantization of a function of p , $f(q, p) = v(p)$ provided by the weight $\varpi_{a\mathcal{W}}$ is $v(P)$ (in the general sense of pseudo-differential operators produced by Fourier transform).*
(iii) *More generally, the quantization of a separable function $f(q, p) = u(q)v(p)$ provided by the weight $\varpi_{a\mathcal{W}}$ is the integral operator*

$$(7.4) \quad \left(A_{u(q)v(p)}^{a\mathcal{W}} \psi \right) (x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} dx' \hat{v}(x' - x) u(\sqrt{x x'}) \psi(x').$$

- (iv) *In particular, the quantization of $u(q)p^n$, $n \in \mathbb{N}$, yields the symmetric operator,*

$$(7.5) \quad A_{u(q)p^n}^{a\mathcal{W}} = \sum_{k=0}^n \binom{n}{k} (-i)^{n-k} u^{(n-k)}(Q) P^k,$$

and for the dilation,

$$(7.6) \quad A_{qp}^{a\mathcal{W}} = D.$$

Proof. The proof is made through a direct application of Eq. (7.1) and application of elementary distribution theory. \square

Therefore, this affine integral quantization is the exact counterpart of the Weyl-Wigner integral quantization [1] and can be termed as canonical as well. However, the choice of such a procedure would lead to three difficulties, at least,

- (1) Since the quantization of the kinetic energy of the free particle on the half-line is just

$$(7.7) \quad A_{p^2}^{a\mathcal{W}} = P^2,$$

and thus does not produce an essentially self-adjoint operator, the corresponding quantum dynamics depends on boundary conditions at the origin $x = 0$. There exists an irreducible ambiguity since different physics are possible on the quantum level.

(2) No classical singularity is cured on the quantum level since

$$(7.8) \quad A_{u(q)}^{a\mathcal{W}} = u(Q), \quad A_{v(p)}^{a\mathcal{W}} = v(P),$$

a feature of the Weyl-Wigner integral quantization as well.

(3) The semi-classical portraits of quantum operators along Eqs. (5.29) and (5.30) cannot be given a probabilistic interpretation, a feature of the Weyl-Wigner integral quantization as well.

The last point is developed below for rank-one operators $|\psi\rangle\langle\psi|$, i.e. pure states.

7.2. Affine Wigner-like quasi-probability. The affine Wigner-like quasi-probability \mathcal{AW}_ψ corresponding to the state ϕ is the application of the general expression (5.29) to the projector $|\phi\rangle\langle\phi|$:

$$(7.9) \quad \mathcal{AW}_\phi(q, p) := \langle\phi|\mathbf{M}^{a\mathcal{W}}(q, p)|\phi\rangle = 2 \int_0^\infty dx \overline{\phi(x)} e^{ip\left(x - \frac{q^2}{x}\right)} \frac{q}{x} \phi\left(\frac{q^2}{x}\right).$$

Proposition 7.4. *Let us consider a pure state ϕ and the corresponding quasi-probability distribution $\mathcal{AW}_\phi(q, p)$. The latter verifies the following properties.*

(i) *It is real*

$$(7.10) \quad \overline{\mathcal{AW}_\phi(q, p)} = \mathcal{AW}_\phi(q, p).$$

(ii) *It is a quasi-probability,*

$$(7.11) \quad \int_{\Pi_+} \frac{dq dp}{2\pi} \mathcal{AW}_\phi(q, p) = 1.$$

(iii) *It satisfies the correct marginalization with respect to variables q and p respectively,*

$$(7.12) \quad \int_0^\infty \frac{dq}{2\pi} \mathcal{AW}_\phi(q, p) = |\hat{\phi}(p)|^2.$$

$$(7.13) \quad \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \mathcal{AW}_\phi(q, p) = |\phi(q)|^2.$$

Proof. (i) and (ii) are direct consequences of the fact that operator $\mathbf{M}^{a\mathcal{W}}$ is symmetric and unit trace (due to the resolution of the identity). (iii) results from elementary integral calculus on (7.17) through change of variable $q \mapsto q^2/x$ and two Fourier transforms (by considering that the support of $\phi(x)$ is included in the positive half-line). \square

A last result concerns the semi-classical portrait of the operator $A_f^{a\mathcal{W}}$ obtained from the expressions (5.30) and (5.32).

Proposition 7.5. *The map $f(q, p) \mapsto \check{f}(q, p) = \text{Tr} \left(A_f^{a\mathcal{W}} \mathbf{M}^{a\mathcal{W}}(q, p) \right)$ yields the following lower symbol*

$$(7.14) \quad \check{f}(q, p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} du e^{ipu} \frac{\hat{f}_p(q, u)}{\sqrt{1 + \frac{u^2}{4q^2}}} = \frac{2q}{\pi} K_0(2qp) *_p f(q, p),$$

where K_0 is a modified Bessel function of the second kind, and $*_p$ is the convolution product with respect to the variable p .

In particular, the lower symbol of the quantization of a function of q alone, $f(q, p) = u(q)$, is $u(q)$.

Thus, there is no “round trip” $f(q, p) \mapsto A_f^{aW} \mapsto \check{f}(q, p) = f(q, p)$ here, contrary to the Wigner map based on the Weyl-Heisenberg symmetry. It is interesting to notice that whereas the lower symbol of $A_p^{aW} = P$ is exactly $\check{p} = p$, the lower symbol of the $A_{p^2}^{aW} = P^2$ is $\check{p}^2 = p^2 + 1/(4q^2)$. Therefore, even though this affine Wigner quantization does not regularizes the kinetic energy on the quantum level, it does on the level of the Wigner quasi-distribution.

7.3. Application to the half-oscillator. We consider the example of the half-harmonic oscillator [32], that is, whose the motion is restricted to the half-line. A physical interpretation of this could be a spring that can be stretched from its equilibrium position but not compressed. In this case where, for convenience, we put $m = 1$ (mass), $\omega = 1$ (frequency), $\hbar = 1$, the affine-Wigner quantization of the classical Hamiltonian $h_{1/2\text{osc}}(q, p) = (p^2 + q^2)/2$ yields $A_h^{aW} = (P^2 + Q^2)/2 \equiv H_{1/2\text{osc}}$, which acts in $L^2(R_+, dx)$ as the Shrödinger operator

$$(7.15) \quad H_{1/2\text{osc}} \phi(x) := \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{2} \right) \phi(x), \quad x > 0.$$

This operator is symmetric but not essentially self-adjoint. In solving the eigenvalue problem $H_{\text{osc}} \phi = E \phi$, it is necessary to choose Dirichlet boundary conditions such that the allowed solutions ϕ in $L^2(R_+)$ satisfy $\phi(0) = 0$. This yields the odd Hermite functions as allowed eigenfunctions, that is, the normalized

$$(7.16) \quad \phi_n(x) = \pi^{-1/4} \frac{1}{2^n \sqrt{(2n-1)!}} H_{2n-1}(x) e^{-\frac{x^2}{2}}$$

for $n = 1, 2, 3, \dots$. As $\phi_n(x)$ is real the corresponding Wigner quasi-density is reduced to the simpler expression:

$$(7.17) \quad \mathcal{AW}_\phi(q, p) = 2 \int_0^\infty dx \phi(x) \cos \left(p \left(x - \frac{q^2}{x} \right) \right) \frac{q}{x} \phi \left(\frac{q^2}{x} \right).$$

In Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10, we give the 2D and 3D plots of the first four eigenfunctions together with their corresponding Wigner quasi-densities $\mathcal{AW}_\phi(q, p)$, their ACS symbols and probability densities defined in terms of thermal states at $t = 0$, i.e. ACS, and for $\alpha = 1$, that is

$$(7.18) \quad \begin{aligned} \text{Tr} \left(\rho_0^{(1)}(q, p) |\phi\rangle \langle \phi| \right) &= \langle \phi | U(q, p) \rho_0^{(1)} U(q, p)^\dagger | \phi \rangle \\ &= \left| \sqrt{\frac{\pi}{q}} \int_0^{+\infty} dx e^{(-\frac{1}{2q} + ip)x} \left(\frac{x}{q} \right)^{\frac{1}{2}} \phi(x) \right|^2 \\ &\equiv |W_\phi(q, p)|^2 \equiv \rho_\phi(q, p). \end{aligned}$$

where $W_\phi(q, p) = \langle q, p | \phi \rangle$ is the ACS symbol of $|\phi\rangle$, equivalently the so-called *wavelet transform* of ϕ with respect to $e_0^{(2)}$.

We notice the organization in negative and positive parts for $\mathcal{AW}_\phi(q, p)$ to be compared with the positive shape of $\rho_\phi(q, p)$.

8. CONCLUSION

In this paper we have explored the possibilities offered by affine covariant integral quantization beyond the familiar case of affine coherent states. The central object is the weight function $\varpi(q, p)$ on the half-plane and its partial Fourier transform with respect to the momentum variable p . The half-plane itself was viewed here as the phase space for the motion of a point particle on the half-line. Actually it can be viewed as the phase space of a dynamical physical quantity which is positive, for which the value 0 represents a singularity. This is the case for instance in cosmology with the volume of the Universe, the canonical conjugate being the expansion coordinate. It would be highly interesting to find other physical examples really accessible to observations, for instance in condensed matter physics, in order to favor this affine quantization with probabilistic content preferably to the Weyl-Heisenberg canonical quantization or the above affine Wigner one. In our sense, the essential self-adjointness of the quantum kinetic energy is a condition which should be always requested, together with a sound probabilistic interpretation of the semi-classical portraits of quantum operators issued from our quantization procedure.

An interesting problem to be addressed in this perspective is to find the class of weight functions ϖ for which the strength of the inverse square potential appearing in Eq. (5.14) is exactly 3/4, i.e.

$$(8.1) \quad \left[2 + 4 \frac{\Omega'(1)}{\Omega(1)} + \frac{\Omega''(1)}{\Omega(1)} \right] = -\frac{3}{4} \quad \text{with} \quad \Omega(u) = \int_0^{+\infty} \frac{dq}{q} \hat{\varpi}_p(u, -q) ,$$

i.e. the lowest limit value for which unique self-adjointness of the quantum kinetic term $P^2 + K/Q^2$ holds, and to compare that class of ϖ 's with the class of ϖ 's for which M^ϖ is a density operator giving rise to a real probabilistic interpretation.

In our next work, we will develop a similar quantization approach based on the unitary irreducible representation of the two-dimensional similitude group $\text{SIM}(2)$ [33] for which the four-dimensional phase space is $\mathbb{R}^2 \times \mathbb{R}_*^2$, where \mathbb{R}_*^2 is the two-dimensional plane with the origin removed. Consistently to one of the main issues of the present work, our method yields a regularization of the singularity at the origin of the configuration plane on the quantum level and opens interesting opportunities in dealing with physical models presenting such point singularities.

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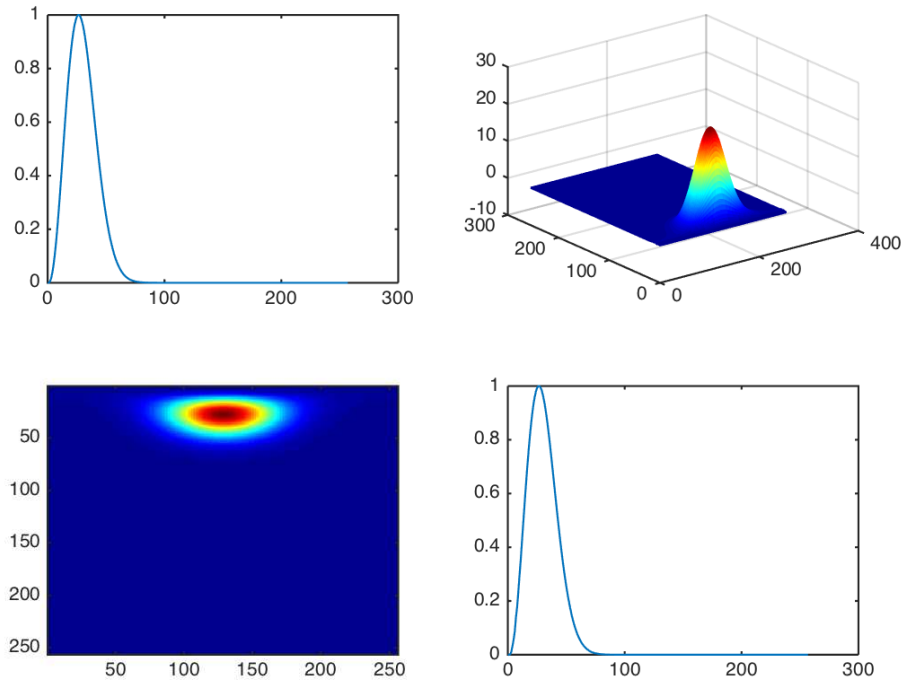


FIGURE 1. First eigenfunction $\phi_1(x) = 2x e^{-\frac{x^2}{2}}$: (a)-Plot of the probability density $|\phi_1(x)|^2$; (b)-3D plot of the corresponding Wigner quasi-density $\mathcal{AW}_{\phi_1}(q, p)$; (c)- 2D plot of $\mathcal{AW}_{\phi_1}(q, p)$; (d)-Sum over p which gives the reconstructed density $|\phi_1(q)|^2$

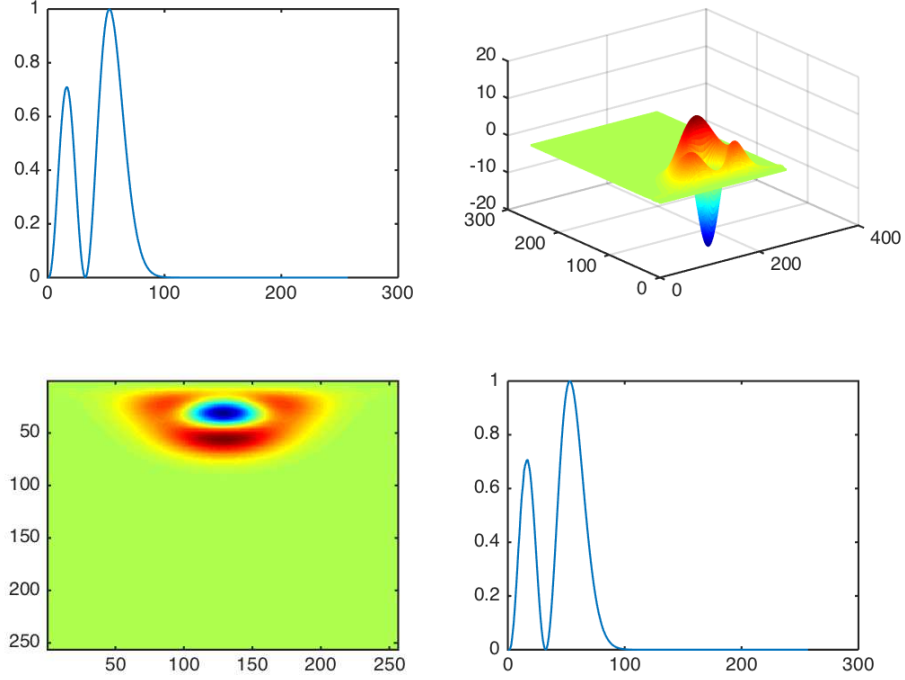


FIGURE 2. Second eigenfunction $\phi_2(x) = 2(3x^2 - 3x) e^{-\frac{x^2}{2}}$: (a)-Plot of the probability density $|\phi_2(x)|^2$; (b)-3D plot of the $\mathcal{AW}_{\phi_2}(q, p)$; (c)-3D plot of $\mathcal{AW}_{\phi_2}(q, p)$; (d)-Sum over p which gives the reconstructed density $|\phi_2(q)|^2$

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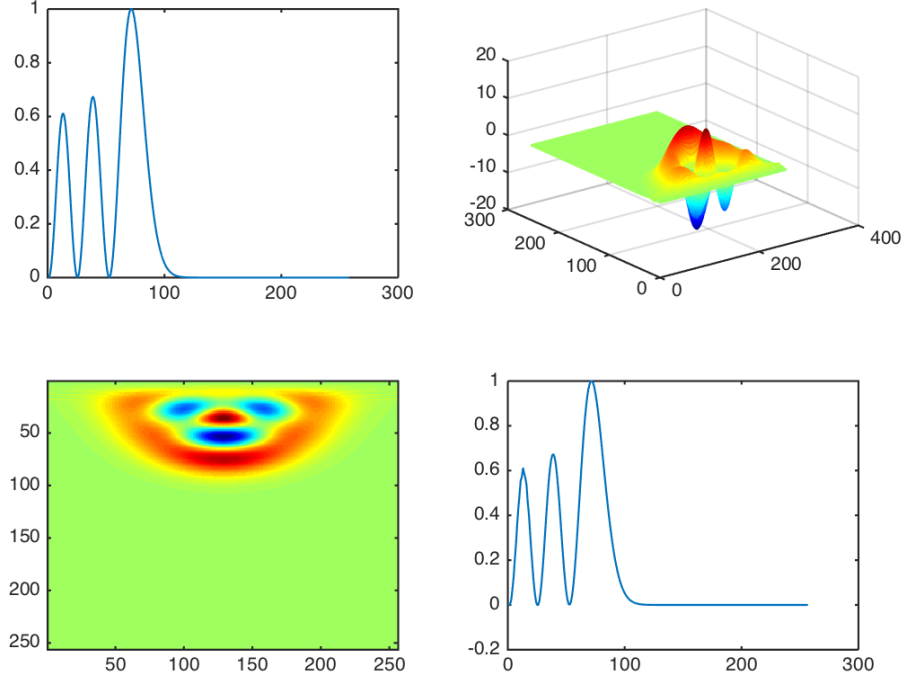


FIGURE 3. Third eigenfunction $\phi_3(x) = 8(4x^5 - 20x^3 + 15x) e^{-\frac{x^2}{2}}$: (a)-Plot of the probability density $|\psi_3(x)|^2$; (b)-3D plot of the $\mathcal{AW}_{\phi_3}(q, p)$; (c)- 3D plot of $\mathcal{AW}_{\phi_3}(q, p)$; (d)-Sum over p which gives the reconstructed density $|\phi_3(q)|^2$

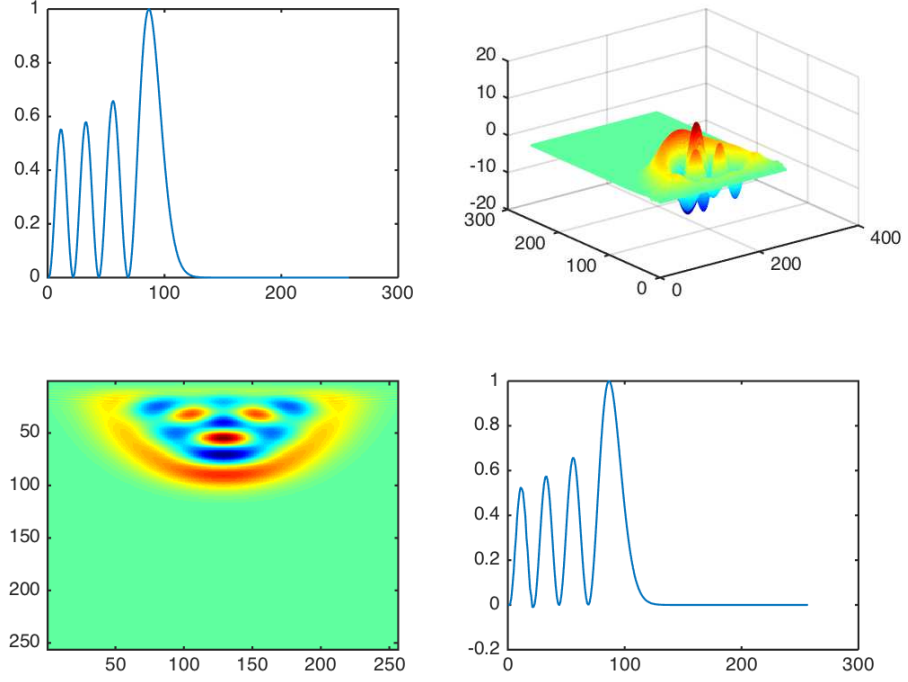


FIGURE 4. Fourth eigenfunction $\phi_4(x) = (128x^7 - 1344x^5 + 3360x^5 + 1334x^3 - 1680x) e^{-\frac{x^2}{2}}$: (a)-Plot of the probability density $|\phi_4(x)|^2$; (b)-3D plot of $\mathcal{AW}_{\phi_4}(q, p)$; (c)- 3D plot of the $\mathcal{AW}_{\psi_4}(q, p)$; (d)-Sum over p which gives the reconstructed density $|\phi_4(q)|^2$

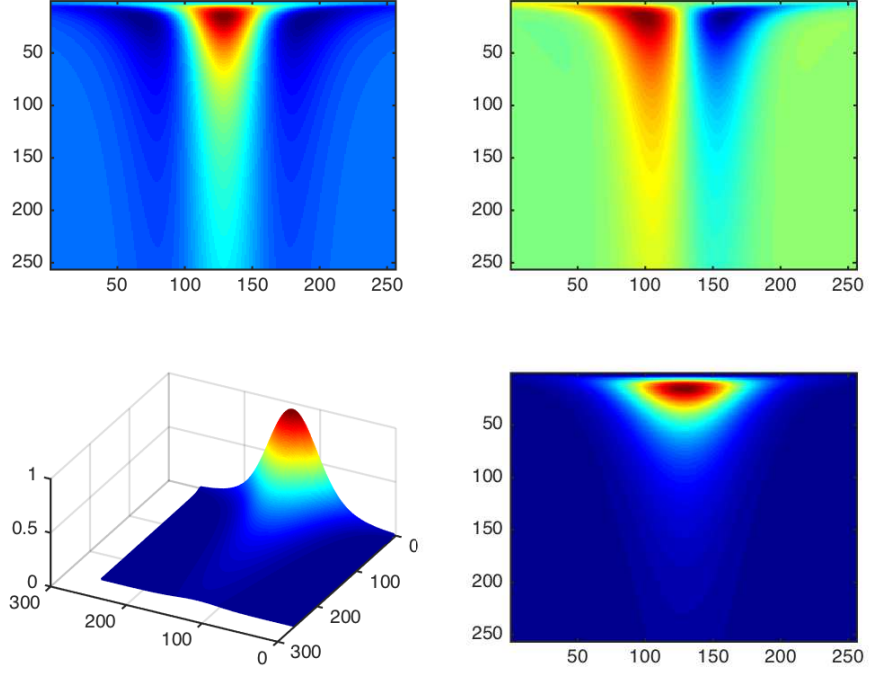


FIGURE 5. First eigenfunction $\phi_1 = 2x e^{-\frac{x^2}{2}}$: (a)-2D plot of the real part of its wavelet transform $W_\phi(q, p)$; (b)- 2D plot of the imaginary part of $W_\phi(q, p)$; (c)- 3D plot of $\rho_\phi(q, p)$ (d)-2D plot of $\rho_\phi(q, p)$.

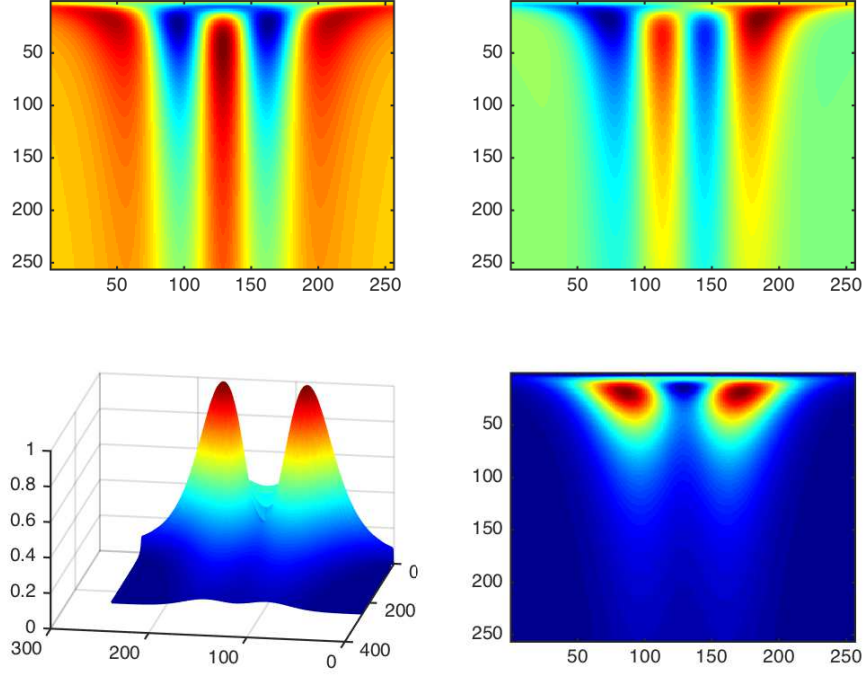


FIGURE 6. Second eigenfunction $\phi_2 = 2(2x^3 - 3x) e^{-\frac{x^2}{2}}$: (a)-2D plot of the real part of $W_\phi(q, p)$; (b)- 2D plot of the imaginary part of $W_\phi(q, p)$; (c)- 3D plot of $\rho_\phi(q, p)$ (d)-2D plot of $\rho_\phi(q, p)$.

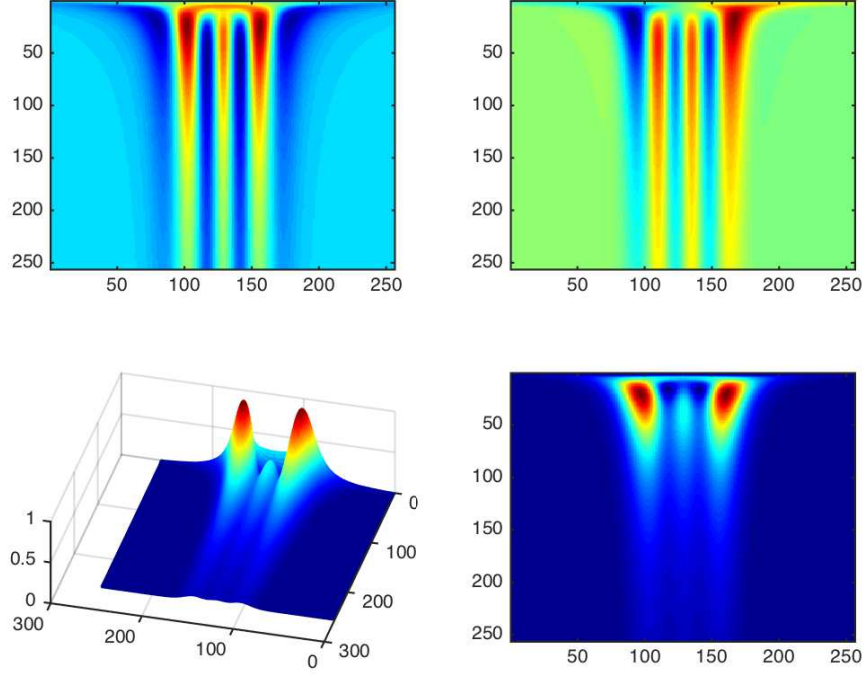


FIGURE 7. Third eigenfunction $\phi_3 = 8(4x^5 - 20x^3 + 15x) e^{-\frac{x^2}{2}}$: (a)- 2D plot of the real part of $W_\phi(q, p)$; (b)- 2D plot of the imaginary part of $W_\phi(q, p)$; (c)- 3D plot of $\rho_\phi(q, p)$ (d)-2D plot of $\rho_\phi(q, p)$.

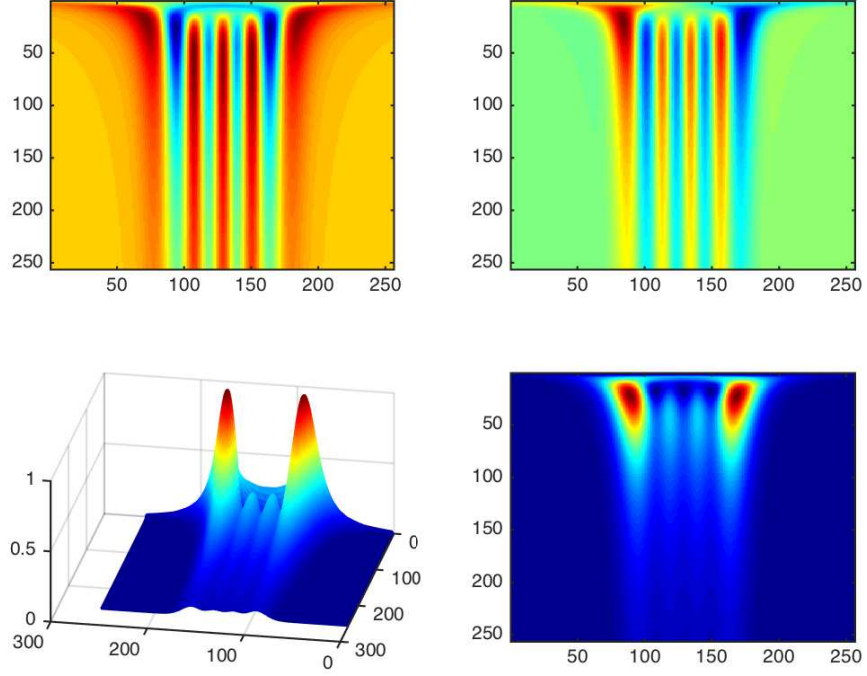


FIGURE 8. Fourth eigenfunction $\phi_4 = (128x^7 - 1344x^5 + 3360x^5 + 1334x^3 - 1680x) e^{-\frac{x^2}{2}}$: (a)-2D plot of the real part of $W_\phi(q, p)$; (b)-2D plot of the imaginary part of $W_\phi(q, p)$; (c)- 3D plot of $\rho_\phi(q, p)$ (d)-2D plot of $\rho_\phi(q, p)$.

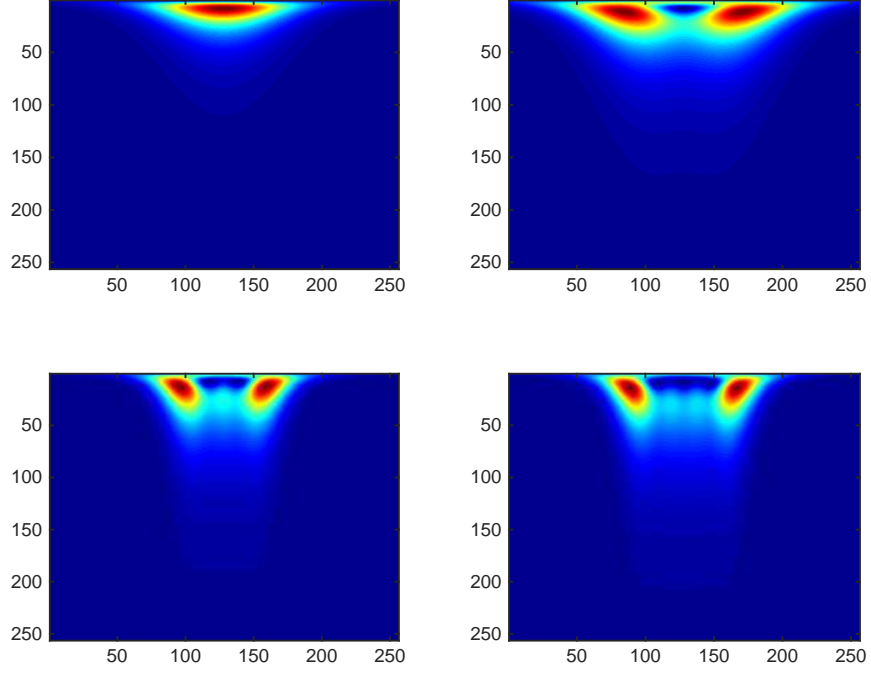


FIGURE 9. 2D plot of $\rho_\phi(q, p)$ for : (a) $\phi(x) = \phi_1(x)$; (b) $\phi(x) = \phi_2(x)$; (c) $\phi(x) = \phi_3(x)$; (d) $\phi(x) = \phi_4(x)$

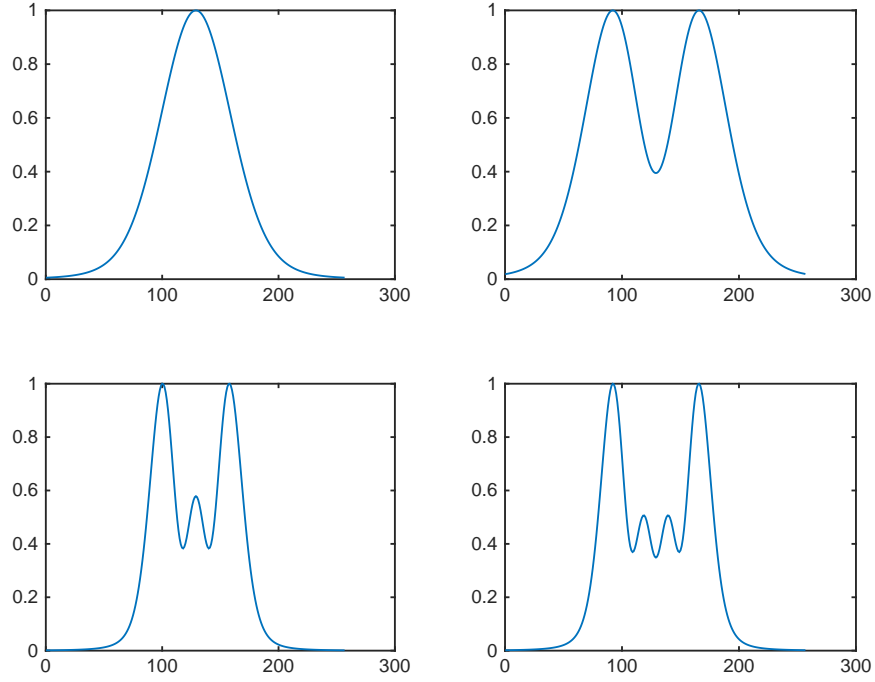


FIGURE 10. 1D plot of the sum over q of $\rho_\phi(q, p)$, i.e, probability density in the momentum space for: (a) $\phi(x) = \phi_1(x)$; (b) $\phi(x) = \phi_2(x)$; (c) $\phi(x) = \phi_3(x)$; (d) $\phi(x) = \phi_4(x)$